# オイラー角から四元数への変換

3 次元直交座標系 xyz の基底ベクトルを次のように表します。

$\underline{\mathrm{x}}_{\mathrm{I}} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ : x 軸方向の単位ベクトル

$\underline{\mathrm{y}}_{\mathrm{I}} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ : y 軸方向の単位ベクトル

$\underline{\mathrm{z}}_{\mathrm{I}} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ : z 軸方向の単位ベクトル

xyz 各軸まわりの回転は、四元数でそれぞれ次のように表されます。

$\mathbf{Q}_{\mathrm{x}} = \cos\frac{\varphi}{2} + \sin\frac{\varphi}{2}\underline{\mathrm{x}}_{\mathrm{I}} = \begin{pmatrix} \cos\frac{\varphi}{2} \\ \sin\frac{\varphi}{2}\underline{\mathrm{x}}_{\mathrm{I}} \end{pmatrix}$ : x 軸まわりの回転を表す四元数

$\mathbf{Q}_{\mathrm{y}} = \cos\frac{\theta}{2} + \sin\frac{\theta}{2}\underline{\mathrm{y}}_{\mathrm{I}} = \begin{pmatrix} \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2}\underline{\mathrm{y}}_{\mathrm{I}} \end{pmatrix}$ : y軸まわりの回転を表す四元数

$\mathbf{Q}_{\mathrm{z}} = \cos\frac{\psi}{2} + \sin\frac{\psi}{2}\underline{\mathrm{z}}_{\mathrm{I}} = \begin{pmatrix} \cos\frac{\psi}{2} \\ \sin\frac{\psi}{2}\underline{\mathrm{z}}_{\mathrm{I}} \end{pmatrix}$ : z軸まわりの回転を表す四元数

基底ベクトルから構成される座標系を、角度ゼロ回転する四元数は、

$\mathbf{Q}_{0} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ : 角度ゼロ回転する四元数

上の四元数が表す座標系を、オイラー角表現で x→y→z の順に回転して新しい直交座標系を表す四元数を作ると、

$$\mathbf{Q}=\mathbf{Q}_0\otimes\mathbf{Q}_x\otimes\mathbf{Q}_y\otimes\mathbf{Q}_z=\mathbf{Q}_x\otimes\mathbf{Q}_y\otimes\mathbf{Q}_z=\begin{pmatrix}\cos\frac{\varphi}{2}\\ \sin\frac{\varphi}{2}\underline{x}_I\end{pmatrix}\otimes\begin{pmatrix}\cos\frac{\theta}{2}\\ \sin\frac{\theta}{2}\underline{y}_I\end{pmatrix}\otimes\begin{pmatrix}\cos\frac{\psi}{2}\\ \sin\frac{\psi}{2}\underline{z}_I\end{pmatrix}$$

$$=\begin{pmatrix}\cos\frac{\varphi}{2}\cos\frac{\theta}{2}-\sin\frac{\varphi}{2}\sin\frac{\theta}{2}\left(\underline{x}_I\cdot\underline{y}_I\right)\\ \sin\frac{\varphi}{2}\cos\frac{\theta}{2}\underline{x}_I+\cos\frac{\varphi}{2}\sin\frac{\theta}{2}\underline{y}_I+\sin\frac{\varphi}{2}\sin\frac{\theta}{2}\left(\underline{x}_I\times\underline{y}_I\right)\end{pmatrix}\otimes\begin{pmatrix}\cos\frac{\psi}{2}\\ \sin\frac{\psi}{2}\underline{z}_I\end{pmatrix}$$

$$=\begin{pmatrix}\cos\frac{\varphi}{2}\cos\frac{\theta}{2}\\ \sin\frac{\varphi}{2}\cos\frac{\theta}{2}\underline{x}_I+\cos\frac{\varphi}{2}\sin\frac{\theta}{2}\underline{y}_I+\sin\frac{\varphi}{2}\sin\frac{\theta}{2}\underline{z}_I\end{pmatrix}\otimes\begin{pmatrix}\cos\frac{\psi}{2}\\ \sin\frac{\psi}{2}\underline{z}_I\end{pmatrix}$$

$$=\begin{pmatrix}\cos\frac{\varphi}{2}\cos\frac{\theta}{2}\cos\frac{\psi}{2}-\sin\frac{\psi}{2}\left(\sin\frac{\varphi}{2}\cos\frac{\theta}{2}\left(\underline{x}_I\cdot\underline{z}_I\right)+\cos\frac{\varphi}{2}\sin\frac{\theta}{2}\left(\underline{y}_I\cdot\underline{z}_I\right)+\sin\frac{\varphi}{2}\sin\frac{\theta}{2}\left(\underline{z}_I\cdot\underline{z}_I\right)\right)\\ \cos\frac{\psi}{2}\left(\sin\frac{\varphi}{2}\cos\frac{\theta}{2}\underline{x}_I+\cos\frac{\varphi}{2}\sin\frac{\theta}{2}\underline{y}_I+\sin\frac{\varphi}{2}\sin\frac{\theta}{2}\underline{z}_I\right)+\cos\frac{\varphi}{2}\cos\frac{\theta}{2}\sin\frac{\psi}{2}\underline{z}_I+\sin\frac{\psi}{2}\left(\sin\frac{\varphi}{2}\cos\frac{\theta}{2}\left(\underline{x}_I\times\underline{z}_I\right)+\cos\frac{\varphi}{2}\sin\frac{\theta}{2}\left(\underline{y}_I\times\underline{z}_I\right)+\sin\frac{\varphi}{2}\sin\frac{\theta}{2}\left(\underline{z}_I\times\underline{z}_I\right)\right)\end{pmatrix}$$

$$=\begin{pmatrix}\cos\frac{\varphi}{2}\cos\frac{\theta}{2}\cos\frac{\psi}{2}-\sin\frac{\psi}{2}\left(\sin\frac{\varphi}{2}\sin\frac{\theta}{2}\right)\\ \cos\frac{\psi}{2}\left(\sin\frac{\varphi}{2}\cos\frac{\theta}{2}\underline{x}_I+\cos\frac{\varphi}{2}\sin\frac{\theta}{2}\underline{y}_I+\sin\frac{\varphi}{2}\sin\frac{\theta}{2}\underline{z}_I\right)+\cos\frac{\varphi}{2}\cos\frac{\theta}{2}\sin\frac{\psi}{2}\underline{z}_I+\sin\frac{\psi}{2}\left(-\sin\frac{\varphi}{2}\cos\frac{\theta}{2}\underline{y}_I+\cos\frac{\varphi}{2}\sin\frac{\theta}{2}\underline{x}_I\right)\end{pmatrix}$$

$$=\begin{pmatrix}\cos\frac{\varphi}{2}\cos\frac{\theta}{2}\cos\frac{\psi}{2}-\sin\frac{\varphi}{2}\sin\frac{\theta}{2}\sin\frac{\psi}{2}\\ \left(\sin\frac{\varphi}{2}\cos\frac{\theta}{2}\cos\frac{\psi}{2}+\cos\frac{\varphi}{2}\sin\frac{\theta}{2}\sin\frac{\psi}{2}\right)\underline{x}_I+\left(\cos\frac{\varphi}{2}\sin\frac{\theta}{2}\cos\frac{\psi}{2}-\sin\frac{\varphi}{2}\cos\frac{\theta}{2}\sin\frac{\psi}{2}\right)\underline{y}_I+\left(\sin\frac{\varphi}{2}\sin\frac{\theta}{2}\cos\frac{\psi}{2}+\cos\frac{\varphi}{2}\cos\frac{\theta}{2}\sin\frac{\psi}{2}\right)\underline{z}_I\end{pmatrix}$$

$$\mathbf{Q} \equiv \begin{pmatrix} q_0 \\ q_1 \\ q_2 \\ q_3 \end{pmatrix} = \begin{pmatrix} \cos\frac{\varphi}{2}\cos\frac{\theta}{2}\cos\frac{\psi}{2} - \sin\frac{\varphi}{2}\sin\frac{\theta}{2}\sin\frac{\psi}{2} \\ \sin\frac{\varphi}{2}\cos\frac{\theta}{2}\cos\frac{\psi}{2} + \cos\frac{\varphi}{2}\sin\frac{\theta}{2}\sin\frac{\psi}{2} \\ \cos\frac{\varphi}{2}\sin\frac{\theta}{2}\cos\frac{\psi}{2} - \sin\frac{\varphi}{2}\cos\frac{\theta}{2}\sin\frac{\psi}{2} \\ \sin\frac{\varphi}{2}\sin\frac{\theta}{2}\cos\frac{\psi}{2} + \cos\frac{\varphi}{2}\cos\frac{\theta}{2}\sin\frac{\psi}{2} \end{pmatrix}$$ :オイラー角 x→y→z に対応した四元数

オイラー角表現で x→y→z の順ではないようなので、逆順にして z→y→x の順に回転して新しい直交座標系を作ると、

$$\mathbf{Q} = \mathbf{Q}_z \otimes \mathbf{Q}_y \otimes \mathbf{Q}_x = \begin{pmatrix} \cos\frac{\psi}{2} \\ \sin\frac{\psi}{2}\underline{z}_I \end{pmatrix} \otimes \begin{pmatrix} \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2}\underline{y}_I \end{pmatrix} \otimes \begin{pmatrix} \cos\frac{\varphi}{2} \\ \sin\frac{\varphi}{2}\underline{x}_I \end{pmatrix}$$

$$= \begin{pmatrix} \cos\frac{\theta}{2}\cos\frac{\psi}{2} - \sin\frac{\theta}{2}\sin\frac{\psi}{2}\left(\underline{z}_I \cdot \underline{y}_I\right) \\ \cos\frac{\theta}{2}\sin\frac{\psi}{2}\underline{z}_I + \sin\frac{\theta}{2}\cos\frac{\psi}{2}\underline{y}_I + \sin\frac{\theta}{2}\sin\frac{\psi}{2}\left(\underline{z}_I \times \underline{y}_I\right) \end{pmatrix} \otimes \begin{pmatrix} \cos\frac{\varphi}{2} \\ \sin\frac{\varphi}{2}\underline{x}_I \end{pmatrix}$$

$$= \begin{pmatrix} \cos\frac{\theta}{2}\cos\frac{\psi}{2} \\ \cos\frac{\theta}{2}\sin\frac{\psi}{2}\underline{z}_I + \sin\frac{\theta}{2}\cos\frac{\psi}{2}\underline{y}_I - \sin\frac{\theta}{2}\sin\frac{\psi}{2}\underline{x}_I \end{pmatrix} \otimes \begin{pmatrix} \cos\frac{\varphi}{2} \\ \sin\frac{\varphi}{2}\underline{x}_I \end{pmatrix}$$

$$= \begin{pmatrix} \cos\frac{\varphi}{2}\cos\frac{\theta}{2}\cos\frac{\psi}{2} - \sin\frac{\varphi}{2}\left(\cos\frac{\theta}{2}\sin\frac{\psi}{2}\left(\underline{z}_I \cdot \underline{x}_I\right) + \sin\frac{\theta}{2}\cos\frac{\psi}{2}\left(\underline{y}_I \cdot \underline{x}_I\right) - \sin\frac{\theta}{2}\sin\frac{\psi}{2}\left(\underline{x}_I \cdot \underline{x}_I\right)\right) \\ \cos\frac{\varphi}{2}\left(\cos\frac{\theta}{2}\sin\frac{\psi}{2}\underline{z}_I + \sin\frac{\theta}{2}\cos\frac{\psi}{2}\underline{y}_I - \sin\frac{\theta}{2}\sin\frac{\psi}{2}\underline{x}_I\right) + \sin\frac{\varphi}{2}\cos\frac{\theta}{2}\cos\frac{\psi}{2}\underline{x}_I + \sin\frac{\varphi}{2}\left(\cos\frac{\theta}{2}\sin\frac{\psi}{2}\left(\underline{z}_I \times \underline{x}_I\right) + \sin\frac{\theta}{2}\cos\frac{\psi}{2}\left(\underline{y}_I \times \underline{x}_I\right) - \sin\frac{\theta}{2}\sin\frac{\psi}{2}\left(\underline{x}_I \times \underline{x}_I\right)\right) \end{pmatrix}$$

$$=\begin{pmatrix} \cos\frac{\varphi}{2}\cos\frac{\theta}{2}\cos\frac{\psi}{2}+\sin\frac{\varphi}{2}\sin\frac{\theta}{2}\sin\frac{\psi}{2} \\ \cos\frac{\varphi}{2}\left(\cos\frac{\theta}{2}\sin\frac{\psi}{2}\underline{z}_I+\sin\frac{\theta}{2}\cos\frac{\psi}{2}\underline{y}_I-\sin\frac{\theta}{2}\sin\frac{\psi}{2}\underline{x}_I\right)+\sin\frac{\varphi}{2}\cos\frac{\theta}{2}\cos\frac{\psi}{2}\underline{x}_I+\sin\frac{\varphi}{2}\left(\cos\frac{\theta}{2}\sin\frac{\psi}{2}\underline{y}_I-\sin\frac{\theta}{2}\cos\frac{\psi}{2}\underline{z}_I\right) \end{pmatrix}$$

$$=\begin{pmatrix} \cos\frac{\varphi}{2}\cos\frac{\theta}{2}\cos\frac{\psi}{2}+\sin\frac{\varphi}{2}\sin\frac{\theta}{2}\sin\frac{\psi}{2} \\ \left(\sin\frac{\varphi}{2}\cos\frac{\theta}{2}\cos\frac{\psi}{2}-\cos\frac{\varphi}{2}\sin\frac{\theta}{2}\sin\frac{\psi}{2}\right)\underline{x}_I+\left(\cos\frac{\varphi}{2}\sin\frac{\theta}{2}\cos\frac{\psi}{2}+\sin\frac{\varphi}{2}\cos\frac{\theta}{2}\sin\frac{\psi}{2}\right)\underline{y}_I+\left(\cos\frac{\varphi}{2}\cos\frac{\theta}{2}\sin\frac{\psi}{2}-\sin\frac{\varphi}{2}\sin\frac{\theta}{2}\cos\frac{\psi}{2}\right)\underline{z}_I \end{pmatrix}$$

$$\mathbf{Q}\equiv\begin{pmatrix} q_0 \\ q_1 \\ q_2 \\ q_3 \end{pmatrix}=\begin{pmatrix} \cos\frac{\varphi}{2}\cos\frac{\theta}{2}\cos\frac{\psi}{2}+\sin\frac{\varphi}{2}\sin\frac{\theta}{2}\sin\frac{\psi}{2} \\ \sin\frac{\varphi}{2}\cos\frac{\theta}{2}\cos\frac{\psi}{2}-cos\frac{\varphi}{2}sin\frac{\theta}{2}\sin\frac{\psi}{2} \\ \cos\frac{\varphi}{2}\sin\frac{\theta}{2}\cos\frac{\psi}{2}+\sin\frac{\varphi}{2}\cos\frac{\theta}{2}\sin\frac{\psi}{2} \\ \cos\frac{\varphi}{2}\cos\frac{\theta}{2}\sin\frac{\psi}{2}-\sin\frac{\varphi}{2}\sin\frac{\theta}{2}\cos\frac{\psi}{2} \end{pmatrix}$$ :オイラー角 z→y→x に対応した四元数

オイラー角表現で z→x→y の順にすると、

$$\mathbf{Q}=\mathbf{Q}_z\otimes\mathbf{Q}_x\otimes\mathbf{Q}_y=\begin{pmatrix} \cos\frac{\psi}{2} \\ \sin\frac{\psi}{2}\underline{z}_I \end{pmatrix}\otimes\begin{pmatrix} \cos\frac{\varphi}{2} \\ \sin\frac{\varphi}{2}\underline{x}_I \end{pmatrix}\otimes\begin{pmatrix} \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2}\underline{y}_I \end{pmatrix}$$

$$=\begin{pmatrix} \cos\frac{\varphi}{2}\cos\frac{\psi}{2}-\sin\frac{\varphi}{2}\sin\frac{\psi}{2}\left(\underline{z}_I\cdot\underline{x}_I\right) \\ \cos\frac{\varphi}{2}\sin\frac{\psi}{2}\underline{z}_I+\sin\frac{\varphi}{2}\cos\frac{\psi}{2}\underline{x}_I+\sin\frac{\varphi}{2}\sin\frac{\psi}{2}\left(\underline{z}_I\times\underline{x}_I\right) \end{pmatrix}\otimes\begin{pmatrix} \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2}\underline{y}_I \end{pmatrix}$$

$$
=\begin{pmatrix}\cos\frac{\varphi}{2}\cos\frac{\psi}{2}\\ \cos\frac{\varphi}{2}\sin\frac{\psi}{2}\underline{z}_I+\sin\frac{\varphi}{2}\cos\frac{\psi}{2}\underline{x}_I+\sin\frac{\varphi}{2}\sin\frac{\psi}{2}\underline{y}_I\end{pmatrix}\otimes\begin{pmatrix}\cos\frac{\theta}{2}\\ \sin\frac{\theta}{2}\underline{y}_I\end{pmatrix}
$$

$$
=\begin{pmatrix}\cos\frac{\varphi}{2}\cos\frac{\theta}{2}\cos\frac{\psi}{2}-\sin\frac{\theta}{2}\left(\cos\frac{\varphi}{2}\sin\frac{\psi}{2}\left(\underline{z}_I\cdot\underline{y}_I\right)+\sin\frac{\varphi}{2}\cos\frac{\psi}{2}\left(\underline{x}_I\cdot\underline{y}_I\right)+\sin\frac{\varphi}{2}\sin\frac{\psi}{2}\left(\underline{y}_I\cdot\underline{y}_I\right)\right)\\ \cos\frac{\theta}{2}\left(\cos\frac{\varphi}{2}\sin\frac{\psi}{2}\underline{z}_I+\sin\frac{\varphi}{2}\cos\frac{\psi}{2}\underline{x}_I+\sin\frac{\varphi}{2}\sin\frac{\psi}{2}\underline{y}_I\right)+\cos\frac{\varphi}{2}\sin\frac{\theta}{2}\cos\frac{\psi}{2}\underline{y}_I+\sin\frac{\theta}{2}\left(\cos\frac{\varphi}{2}\sin\frac{\psi}{2}\left(\underline{z}_I\times\underline{y}_I\right)+\sin\frac{\varphi}{2}\cos\frac{\psi}{2}\left(\underline{x}_I\times\underline{y}_I\right)+\sin\frac{\varphi}{2}\sin\frac{\psi}{2}\left(\underline{y}_I\times\underline{y}_I\right)\right)\end{pmatrix}
$$

$$
=\begin{pmatrix}\cos\frac{\varphi}{2}\cos\frac{\theta}{2}\cos\frac{\psi}{2}-\sin\frac{\varphi}{2}\sin\frac{\theta}{2}\sin\frac{\psi}{2}\\ \cos\frac{\theta}{2}\left(\cos\frac{\varphi}{2}\sin\frac{\psi}{2}\underline{z}_I+\sin\frac{\varphi}{2}\cos\frac{\psi}{2}\underline{x}_I+\sin\frac{\varphi}{2}\sin\frac{\psi}{2}\underline{y}_I\right)+\cos\frac{\varphi}{2}\sin\frac{\theta}{2}\cos\frac{\psi}{2}\underline{y}_I+\sin\frac{\theta}{2}\left(-\cos\frac{\varphi}{2}\sin\frac{\psi}{2}\underline{x}_I+\sin\frac{\varphi}{2}\cos\frac{\psi}{2}\underline{z}_I\right)\end{pmatrix}
$$

$$
=\begin{pmatrix}\cos\frac{\varphi}{2}\cos\frac{\theta}{2}\cos\frac{\psi}{2}-\sin\frac{\varphi}{2}\sin\frac{\theta}{2}\sin\frac{\psi}{2}\\ \left(\sin\frac{\varphi}{2}\cos\frac{\theta}{2}\cos\frac{\psi}{2}-\cos\frac{\varphi}{2}\sin\frac{\theta}{2}\sin\frac{\psi}{2}\right)\underline{x}_I+\left(\sin\frac{\varphi}{2}\cos\frac{\theta}{2}\sin\frac{\psi}{2}+\cos\frac{\varphi}{2}\sin\frac{\theta}{2}\cos\frac{\psi}{2}\right)\underline{y}_I+\left(\cos\frac{\varphi}{2}\cos\frac{\theta}{2}\sin\frac{\psi}{2}+\sin\frac{\varphi}{2}\sin\frac{\theta}{2}\cos\frac{\psi}{2}\right)\underline{z}_I\end{pmatrix}
$$

$$
\mathbf{Q}\equiv\begin{pmatrix}q_0\\q_1\\q_2\\q_3\end{pmatrix}=\begin{pmatrix}\cos\frac{\varphi}{2}\cos\frac{\theta}{2}\cos\frac{\psi}{2}-\sin\frac{\varphi}{2}\sin\frac{\theta}{2}\sin\frac{\psi}{2}\\ \sin\frac{\varphi}{2}\cos\frac{\theta}{2}\cos\frac{\psi}{2}-\cos\frac{\varphi}{2}\sin\frac{\theta}{2}\sin\frac{\psi}{2}\\ \sin\frac{\varphi}{2}\cos\frac{\theta}{2}\sin\frac{\psi}{2}+\cos\frac{\varphi}{2}\sin\frac{\theta}{2}\cos\frac{\psi}{2}\\ \cos\frac{\varphi}{2}\cos\frac{\theta}{2}\sin\frac{\psi}{2}+\sin\frac{\varphi}{2}\sin\frac{\theta}{2}\cos\frac{\psi}{2}\end{pmatrix}
$$

:オイラー角 z→x→y に対応した四元数